Dispersion, spreading and sparsity of Gabor wave packets

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Joint work with Elena Cordero & Fabio Nicola

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Gabor wave packets

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- $\pi(x,\xi) = M_\xi T_x$ is the time-frequency shift along $(x,\xi)\in\mathbb{R}^{2d}$, with

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- $g \in \mathcal{S}(\mathbb{R}^d)$ is a function possessing good localization in phase space, that is g and its Fourier transform $\hat{g}$ are in some sense concentrated in small sets $T \in \mathbb{R}^d$ and $\Omega \in \widehat{\mathbb{R}^d}$ respectively.

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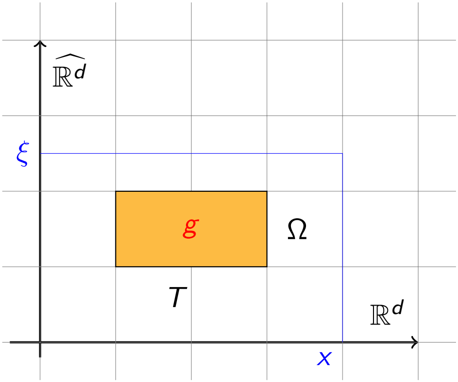

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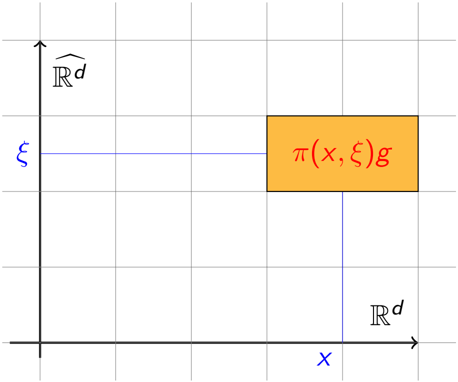

Gabor analysis of functions

Decomposition of $f \in \mathcal{S}'(\mathbb{R}^d)$ along Gabor wave packets (Gabor/short-time Fourier transform):

$$V_g f(x,\xi) := \langle f, \pi(x,\xi)g \rangle = \int_{\mathbb{R}^d} e^{-2\pi i y\cdot\xi} f(y)\,\overline{g(y-x)}\,dy, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

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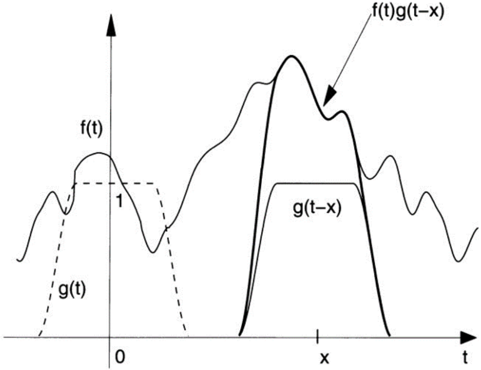

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Modulation spaces are Banach spaces containing functions with prescribed phase-space integrability: for $1 \le p \le \infty$ and $s \in \mathbb{R}$,

$$M^p_{v_s}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{M^p_{v_s}} < \infty \right\},$$

where (for $1 \le p < \infty$ - and similarly for $p = \infty$)

$$\|f\|_{M^p_{v_s}} := \left(\int_{\mathbb{R}^{2d}} |V_g f(z)|^p (1+|z|)^{sp} dz \right)^{1/p}.$$

Gabor analysis of operators

The Gabor transform allows one to perform phase-space analysis of a linear continuous operator $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$:

$$V_\gamma(Af)(w) = \int_{\mathbb{R}^{2d}} K_A(w,z) V_g f(z) dz, \quad w \in \mathbb{R}^{2d},$$

where we introduced the Gabor matrix/kernel of A with respect to the windows $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ (with $\|g\|_{L^2} = \|\gamma\|_{L^2} = 1$):

$$K_A(w,z) := \langle A\pi(z)g, \pi(w)\gamma \rangle, \quad w, z \in \mathbb{R}^{2d}.$$

Several results have been appearing in the literature for pseudodifferential operators, Fourier integral operators and propagators associated with Schrödinger-type evolution equations.

Metaplectic operators

Let $S \in \mathrm{Sp}(d, \mathbb{R})$ be a symplectic matrix, that is

$$S^\top J S = J, \qquad J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}.$$

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There exists a double-valued unitary representation μ of $\mathrm{Sp}(d,\mathbb{R})$ on $L^2(\mathbb{R}^d)$, called the metaplectic representation, such that the metaplectic operator $\mu(S)$ satisfies the intertwining relation

$$\pi(Sz) = \mu(S)\pi(z)\mu(S)^{-1}, \quad z \in \mathbb{R}^{2d}.$$

Metaplectic operators and Schrödinger propagators

Let Q be a real quadratic form on $\mathbb{R}^{2d}$, namely

$$Q(x,\xi) = \frac{1}{2}\xi \cdot A\xi + \xi \cdot Bx + \frac{1}{2}x \cdot Cx, \quad A, C \in \mathbb{R}^{d\times d}_{\mathrm{sym}},$$

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and consider its Weyl quantization:

$$Q^{\mathrm{w}} = -\frac{1}{8\pi^2}\sum_{j,k=1}^{d} A_{j,k}\partial^2_{j,k} - \frac{i}{2\pi}\sum_{j,k=1}^{d} B_{j,k}x_j\partial_k - \frac{1}{4\pi}\mathrm{Tr}(B) + \frac{1}{2}\sum_{j,k=1}^{d} C_{j,k}x_jx_k.$$

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The propagator for the Schrödinger equation $i\partial_t\psi = 2\pi Q^{\mathrm{w}}\psi$ is a metaplectic operator, that is

$$U(t) = e^{-2\pi i t Q^{\mathrm{w}}} = \pm c(t)\mu(S_t),$$

where $c(t) \in \mathbb{C}$, $|c(t)| = 1$ and $t \mapsto S_t \in \mathrm{Sp}(d,\mathbb{R})$ is the solution of the classical equations of motion with Hamiltonian $Q(x,\xi)$ in phase space.

Gabor analysis of the Schrödinger propagator

Consider the Schrödinger propagator for the free particle $U(t) = e^{i(t/2\pi)\triangle}$, $t \in \mathbb{R}$, and fix $g \in \mathcal{S}(\mathbb{R}^d)$. Then

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For any $t \in \mathbb{R}$ and $N \in \mathbb{N}$ there exists a constant $C = C(t, N) > 0$ such that

$$|\langle e^{i(t/2\pi)\triangle}\pi(z)g, \pi(w)g\rangle| \leq C(1 + |w - S_t z|)^{-N}, \quad w, z \in \mathbb{R}^{2d}.$$

Sparsity phenomenon: the phase-space representation of $U(t)$ is essentially concentrated along the graph of the classical flow S_t - in according with the correspondence principle of quantum mechanics.

The correspondence principle...

Initial datum: $\psi_0(y) = M_{\xi_0} T_{x_0} e^{-\pi|y|^2} = e^{2\pi i \xi_0 \cdot y} e^{-\pi|y-x_0|^2}$.
Phase-space effect of $U(t)$: the concentration is approximately moved along the classical flow $x(t) = x_0 + 2t\xi_0$, $\xi(t) = \xi_0$.

(Simulation with $x_0 = 10$ and $\xi_0 = 1$.)

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The problem

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Is there a way to provide refined estimates for the Gabor kernel of a metaplectic operator $\mu(S)$ where all these features are fully and simultaneously represented?

The Euler decomposition of a symplectic matrix

For any $S \in \mathrm{Sp}(d, \mathbb{R})$ there exist (non-unique) *symplectic rotations* $U, V \in \mathrm{Sp}(d, \mathbb{R}) \cap \mathrm{O}(2d, \mathbb{R})$ such that

$$S = U^\top D V, \quad D = \begin{bmatrix} \Sigma & O \\ O & \Sigma^{-1} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_d \end{bmatrix},$$

where $\sigma_1 \geq \ldots \geq \sigma_d \geq 1 \geq \sigma_d^{-1} \geq \ldots \geq \sigma_1^{-1} > 0$ are the singular values of S.

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We also introduce the related matrix

$$D' = \begin{bmatrix} \Sigma^{-1} & O \\ O & I \end{bmatrix}.$$

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Theorem (CNT 2020 - Gabor atoms in the Schwartz class)

For any $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$ *and* $N > 0$ *there exists* $C > 0$ *such that, for every* $S \in \mathrm{Sp}(d, \mathbb{R})$ *and any Euler decomposition of* S*,*

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma \rangle| \le C(\det \Sigma)^{-1/2}(1 + |D'U(w - Sz)|)^{-N},$$

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for any $z, w \in \mathbb{R}^{2d}$.

Note that all the expected features of the Gabor kernel are simultaneously represented here, in particular:

- dispersion - multiplication by $(\det \Sigma)^{-1/2}$;
- spreading - dilation by $D'U$;
- sparsity - quasi-diagonal structure along S.

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Back to the Schrödinger propagator: sparsity and dispersion

Recall that $e^{i(t/2\pi)\triangle} = \pm c(t)\mu(S_t)$, where $S_t = \begin{bmatrix} I & 2tI \\ O & I \end{bmatrix}$.

Back to the Schrödinger propagator: sparsity and dispersion

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$$|\langle e^{i(t/2\pi)\triangle}\pi(z)g, \pi(w)\gamma\rangle| \leq C(\det \Sigma_t)^{-1/2}(1 + |D_t' U_t(w - S_t z)|)^{-N}.$$

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The largest d singular values of S_t coincide:

$$\sigma_j = \sigma(t) = \sqrt{1+t^2} + |t|, \quad j = 1, \ldots, d,$$

hence $(\det \Sigma_t)^{-1/2} \asymp (1 + |t|)^{-d/2}$ as expected.

Back to the Scrödinger propagator: spreading

The spreading phenomenon manifests itself as a dilation by

$$D_t' U_t = \frac{1}{\sqrt{1+\sigma(t)^2}} \begin{bmatrix} I & \sigma(t)^{-1} I \\ -I & \sigma(t) I \end{bmatrix}, \quad t \geq 0.$$

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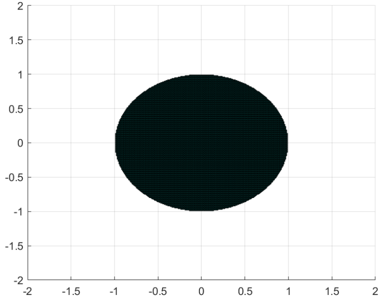
Toy example for $d=1$: wave packet $\pi(z)g$ with $z=0$ and g concentrated on the unit ball S in $\mathbb{R}^2$.



$$(D_t' U_t)^{-1}(S) = \left\{ (x,\xi) \in \mathbb{R}^2 : |D_t' U_t(x,\xi)| < 1 \right\}.$$

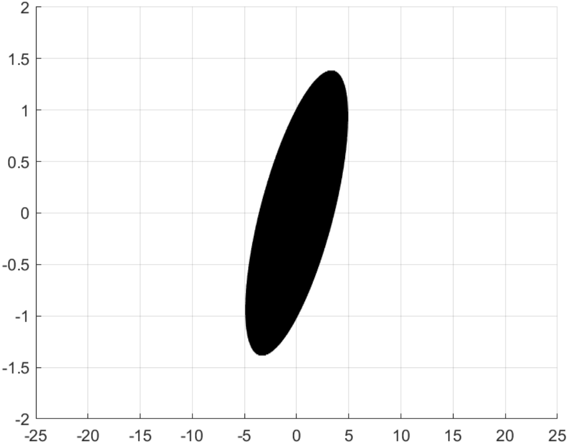


Figure: $(D_t' U_t)^{-1}(S)$ for $t = 2.4$

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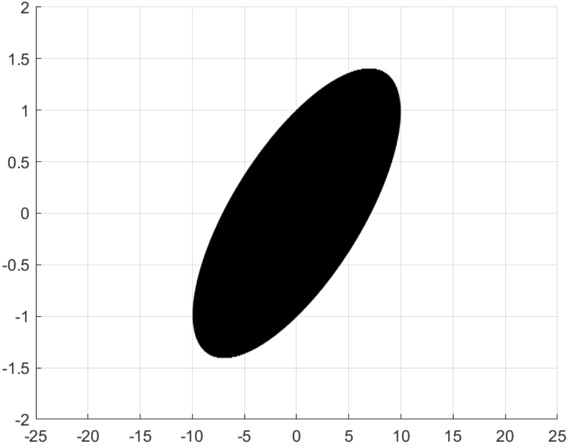

Figure: $(D_t' U_t)^{-1}(S)$ for $t = 5$

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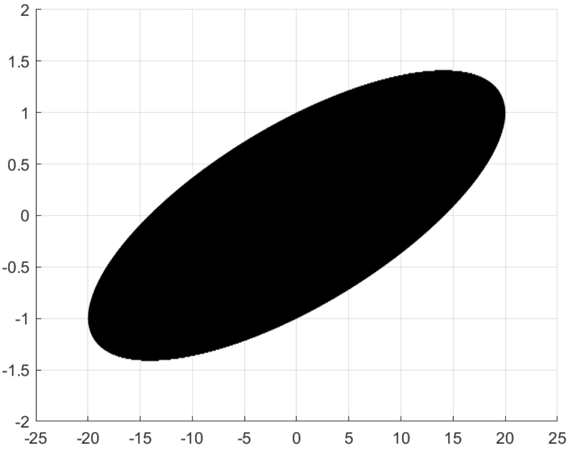

Figure: $(D_t' U_t)^{-1}(S)$ for $t = 10$

Further results

Theorem (CNT 2020 - Gabor atoms in modulation spaces)

1. *Let* $1 \le p, q, r \le \infty$ *satisfy* $1/p + 1/q = 1 + 1/r$. *For any* $g \in M^p(\mathbb{R}^d)$, $\gamma \in M^q(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d, \mathbb{R})$, *there exists* $H \in L^r(\mathbb{R}^{2d})$ *such that, for any* $z, w \in \mathbb{R}^{2d}$,

$$|\langle \mu(S)\pi(z)g, \pi(w)\gamma\rangle| \le H(D'U(w - Sz)), \qquad (\star)$$

with

$$\|H\|_{L^r} \le (\det \Sigma)^{1/2-1/r} \|g\|_{M^p} \|\gamma\|_{M^q}.$$

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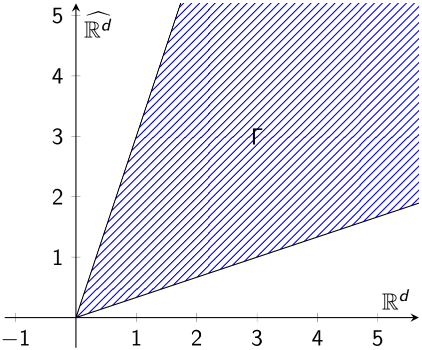
2. *Let* $s > 2d$. *For any* $g, \gamma \in M^\infty_{v_s}(\mathbb{R}^d)\,(\subset M^1(\mathbb{R}^d))$ *there exists* $H \in L^\infty_{v_{s-2d}}(\mathbb{R}^{2d})$ *such that* $(\bigstar)$ *holds, with*

$$\|H\|_{L^\infty_{v_{s-2d}}} \le (\det \Sigma)^{-1/2} \|g\|_{M^\infty_{v_s}} \|\gamma\|_{M^\infty_{v_s}}.$$

Some applications

Given an open cone Γ in $\mathbb{R}^{2d}$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ we define the space of $M^1_{(g)}(\Gamma)$ of M^1-regular distributions on the cone Γ with respect to g as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

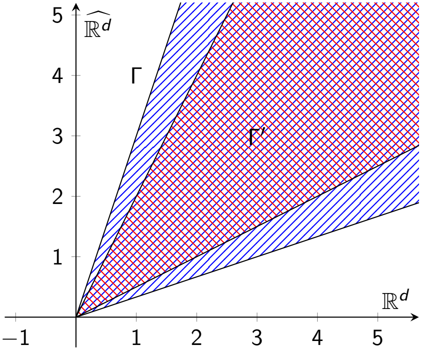
$$\|f\|_{M^1_{(g)}(\Gamma)} := \int_{\Gamma} |V_g f(z)| dz < \infty.$$



Some applications

Theorem (CNT 2020 - M^1 regularity on a cone is preserved by $\mu(S)$)

Let $S \in \mathrm{Sp}(d,\mathbb{R})$, $g,\gamma \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and $\Gamma,\Gamma' \subset \mathbb{R}^{2d}$ be open cones such that $\overline{\Gamma' \cap \mathbb{S}^{2d-1}} \subset \Gamma \cap \mathbb{S}^{2d-1}$. If $f \in \mathcal{S}'(\mathbb{R}^d)$ is M^1-regular on Γ with respect to g then $\mu(S)f$ is M^1-regular on $S(\Gamma')$ with respect to γ.



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Corollary

Consider $1 \leq p \leq \infty$. *There exists* $C > 0$ *such that, for any* $f \in M^p(\mathbb{R}^d)$, $S \in \mathrm{Sp}(d,\mathbb{R})$,

$$\|\mu(S)f\|_{M^p} \leq C(\det \Sigma)^{|1/2-1/p|}\|f\|_{M^p}.$$

Want more details?

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